MATH 54, mock final test.

All the necessary work to justify an answer and all the necessary steps of a proof must be shown clearly to obtain full credit. Partial credit **may** be given but only for significant progress towards a solution. Show all relevant work in logical sequence and indicate all answers clearly. Cross out all work you do not wish considered. 2 pages of notes are allowed. Books and electronic devices are not allowed during the test.

---

1. Determine for which values of b_1, b_2, b_3 the system

$$\begin{aligned} 2x_1 - 4x_2 - 2x_3 &= b_1 \\ -5x_1 + x_2 + x_3 &= b_2 \\ 7x_1 - 5x_2 - 3x_3 &= b_3 \end{aligned}$$

has a solution.

2. Let $T : \mathbb{P}_2 \to \mathbb{R}^3$ be the linear map taking a polynomial of degree at most 2 to its values at the points -1, 0 and 1:

$$T(p) := \begin{bmatrix} p(-1) \\ p(0) \\ p(1) \end{bmatrix}.$$

(a) Write the matrix of T in the standard bases.

(b) Using the result of (a), find a polynomial $f \in \mathbb{P}_2$ such that $f(-1) = 1$, $f(0) = 0$, $f(1) = 2$.

3. Let S be the vector space of functions of the form

$$S := \{p(x)e^{2x} : p \in \mathbb{P}_2\}$$

and L be the differential operator $L := (D - 2I)^2 = (D - 2I)(D - 2I)$ where I is the identity map and $D = \frac{d}{dx}$ is the operator of differentiation. Is there a basis for S consisting of eigenvectors of L?

4. Using variation of parameters, find a particular solution to the equation

$$y'' + y = \sec t.$$

5. Find a general solution to the equation

$$y^{(4)} + 4y = x^5 + 2x^4 - x^3 + 1.$$

6. Do the functions $\{e^{3x}, e^{-x}, e^{-4x}\}$ form a fundamental solution set for the differential equation

$$y''' + 2y'' - 11y' - 12y = 0?$$

7. Consider the following system of second order differential equations in two functions $y = y(t)$ and $z = z(t)$:

$$\begin{aligned} y'' + 16z &= 0 \\ z'' - y &= 0 \end{aligned}$$

(a) Convert the system to a system of *first order* differential equations in four functions x_1, x_2, x_3, and x_4.

(b) Use part (a) to produce four linearly independent solutions to the original system.

8. Compute the Fourier series of e^x on the interval $[-\pi, \pi]$.

9. (a) Determine all solutions of the form $u(x,t) = X(x)T(t)$ to the heat equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}\ , \qquad 0 < x < \pi, \quad t > 0,$$

that satisfy the boundary conditions

$$u(0,t) = 0, \qquad u(\pi,t) + \frac{\partial u}{\partial x}(\pi,t) = 0, \qquad t > 0.$$

(b) Describe how to obtain, given a function $f(x)$, a solution to the boundary value problem (a) that also satisfies the initial condition

$$u(x,0) = f(x), \qquad 0 < x < \pi.$$